PROPAGATION OF PERTURBATIONS IN A LIQUID CONTAINING GAS BUBBLES

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One of the models of a bubble-containing medium (Iordanskii's system of equations of motion), based on liquid motion "averaged" on the assumption that bubble pulsation conforms to the Lamb equation, is investigated.

Solution of Iordanskii's linearized system gives a relationship between the phase velocity of sound and the plane-wave frequency. An evaluation of this relationship for a particular bubble size distribution agrees with known experimental results.

If the liquid component of the medium is incompressible, Iordanskii's system for bubbles of one kind reduces approximately to a system of two second-order partial differential equations for the pressure and concentration of gas in the medium. A solution of this system is found. For particular relative values of the parameters of the medium (length, gas concentration, and bubble size) the processes of perturbation propagation in bubble-containing media are similar. The similarity criterion is found from the system solution and is confirmed experimentally.

All the known theoretical works devoted to the considered question can be divided into two approaches, each of which has its own model of a bubble-containing medium. Both approaches are based on the "averaged" motion of the liquid containing gas bubbles. The difference is that in one approach [1, 2] the pressure in the gas bubble is always equal to the pressure in the liquid, and in the other [3] the pulsation of the bubbles is given by the Lamb equation. The calculations presented below are based on Iordanskii's equations [3].

These equations in the unidimensional case have the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0,\qquad \frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+\frac{1}{\rho}\,\frac{\partial p}{\partial x}=0,\qquad \rho=\left(\rho_0+\frac{p-p_0}{c_0^2}\right)(1+k)^{-1},$$

$$k=\sum_{j=1}^{N}k_j,\qquad \frac{k_j}{k_{j0}}=\left(\frac{R_j}{R_{j0}}\right)^3,\qquad \rho_0\left(R_j\,\frac{d^2R_j}{dt^2}+\frac{3}{2}\left(\frac{dR_j}{dt}\right)^2\right)=p_0\left(\frac{R_j}{R_{j0}}\right)^{-3\gamma}-p$$

$$p=p_0,\quad k_j=k_{j0},\quad R_j=R_{j0}\ \text{ when } t=0\,. \tag{0.1}$$

Here ρ, p, and u are the averaged density, pressure, and velocity of the particle in the medium and k_j is the volume concentration of gas for bubbles of radius R_j. The following assumptions were made in the determination of system (0.1) in [3]:

1. the characteristic length L of the average motion, the average distance l between the bubbles, and the radius R_j of the bubbles satisfy the inequalities $L \gg l \gg R_j$;

2. nonsphericity of the bubbles and the gas mass can be neglected;

3. the equation of state is written in the acoustic approximation for the liquid component of the medium;

4. the initial values k_{j0}, R_{j0}, and p_0 are independent of x.

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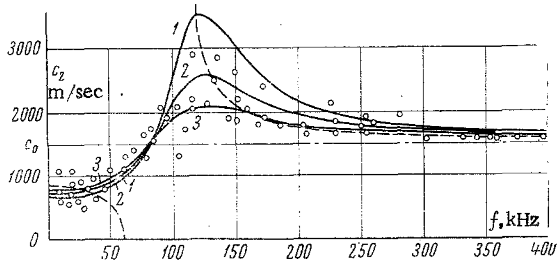

Fig. 1

By comparison with Iordanskii's equations we make one additional assumption here. In the third term of the second equation of (0.1) we omit the term

$$\sum_{j=1}^{N} \rho_0 k_j \left(\frac{dR_j}{dt}\right)^2$$

in the derivative.

1. We determine the velocity of propagation for small perturbations. Linearizing (0.1) and eliminating the density, we obtain

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} - \rho_0 \sum_{j=1}^{N} k_{j0} \frac{\partial^2}{\partial t^2}\frac{k_j}{k_{j0}} = 0,$$
$$\frac{\partial^2}{\partial t^2}\frac{k_j}{k_{j0}} + \Omega_j^2 \frac{k_j}{k_{j0}} = \frac{\Omega_j^2}{\gamma p_0} p \qquad \left(\Omega_j^2 = \frac{3\gamma p_0}{\rho_0 R_{j_0}^2}\right). \tag{1.1}$$

Here Ω_j is the natural frequency of the bubble. We seek the solution in the form

$$p = Ae^{i(\omega t - mx)}, \qquad k_j / k_{j0} = B_j e^{i(\omega t - mx)}.$$

From (1.1) we easily obtain the following relationship for the phase velocity of sound c_2:

$$\frac{c_0^2}{c_2^2} = 1 + \frac{c_0^2}{c_1^2}\sum_{j=1}^{N}\frac{k_{j0}}{k_0}\left(1 - \frac{\omega^2}{\Omega_j^2}\right)^{-1} \qquad \left(c_1^2 = \frac{\gamma p_0}{\rho_0 k_0}\right). \tag{1.2}$$

Here c_0 is the sound velocity in the liquid and c_1 is the sound velocity in the medium according to the equilibrium model [1]. Thus, (1.1) describes the motion with dispersion.

With $N \to \infty$ and with the limit conversion in (1.2) we can write

$$\frac{c_0^2}{c_2^2} = 1 + \frac{c_0^2}{c_1^2}\int_0^\infty \frac{k(R)\,dR}{1 - \omega^2/\Omega^2(R)} \qquad \left(\int_0^\infty k(R)\,dR = 1\right). \tag{1.3}$$

Here k(R) is the fractional concentration of bubbles of a particular kind. The integral in (1.3) for function k(R) of the form

$$\frac{(R/b)^2}{1 + (R/b)^4} \tag{1.4}$$

where b is a scale approximating the experimental bubble size distribution [4], can be determined. In this

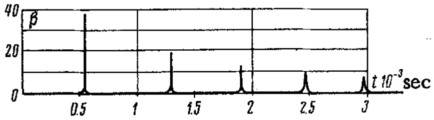

Fig. 2

case the phase velocity is

$$\frac{c_0^2}{c_2^2} = 1 + \frac{c_0^2}{c_1^2}\frac{1-(\omega/\Omega(b))^2}{1+(\omega/\Omega(b))^4} \qquad \left(\Omega^2(b) = \frac{3\gamma p_0}{\rho_0 b^2}\right). \tag{1.5}$$

Here $\Omega(b)$ is the scale frequency.

Figure 1 shows the results of calculating the relationship between the phase velocity c_2 and the frequency f ($f=\omega/2\pi$) from Eq. (1.5) with the following initial parameters: p_0=1 atm, γ=1.4, k_0=0.00025, 0.00020, and 0.00015 (curves 1, 2, and 3, respectively). The results of [4] are represented by experimental points demarcating the region of spread for the experimental data. Despite the great spread of the experimental data (this can presumably be attributed to the instability of the concentration, which varied in the range 0.00015-0.00025), it is easy to follow the general nature of the variation of the phase velocity with frequency.

The calculation was carried out for three concentrations so that a spread of calculated values corresponding to the experimental spread would be obtained. The scale b was chosen so that the calculated data from (1.5) would agree with the experimental data at the point $c_2=c_0$ (in Fig. 1, f=80 kHz corresponds to this point). The broken line is the calculated relationship for the actual phase velocity for a medium composed of equal bubbles with radius R_0 = 0.0055 cm, corresponding to the largest "partial" concentration k_0=0.00015. The agreement between calculated curves and experimental data is satisfactory.

Figure 1 shows that $c_2 \to c_0$ when $f \to \infty$; with very long waves ($f \to 0$) the medium attains equilibrium completely, the dispersion disappears, and $c_2 \to c_1$ (when $c_0 \gg c_1$).

From the obtained result we can conclude that (0.1) corresponds to a real flow of liquid containing gas bubbles.

2. We make some estimates of shock-wave interaction in a bubble-containing medium. To do this we transform (0.1) by introducing the following simplifications:

1. we regard the liquid as containing bubbles of only one kind;
2. we assume that $c_0=\infty$;
3. in the first, second, and fourth equations of (0.1) we omit the terms

$$u\frac{\partial p}{\partial x}, \qquad u\frac{\partial u}{\partial x}, \qquad \frac{1}{2}\left(\frac{d}{dt}\frac{R}{R_0}\right)^2.$$

As a result we obtain

$$\frac{\partial^2 p}{\partial x^2} + \rho_0 k_0 \frac{\partial^2}{\partial t^2}\frac{k}{k_0} = 0 \qquad \frac{d^2}{dt^2}\frac{k}{k_0} = \frac{3}{\rho_0 R_0^2}\left(\frac{k}{k_0}\right)^{1/3}\left(p_0\left(\frac{k}{k_0}\right)^{-\gamma} - p\right). \tag{2.1}$$

We introduce the new symbols

$$y = \frac{R}{R_0} = \left(\frac{k}{k_0}\right)^{1/3}, \qquad \zeta = p - p_0 y^{-3\gamma}, \qquad \xi = \left(\frac{d}{dt}\frac{k}{k_0}\right)^2.$$

Regarding x and y as new independent variables, ζ and ξ as the required functions, and assuming $\partial y/\partial x$ and u small, we obtain instead of (2.1)

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{3k_0 y}{R_0^2}\zeta, \qquad \frac{\partial \xi}{\partial y} = -\frac{18 y^3}{\rho_0 R_0^2}\zeta. \tag{2.2}$$

The solution of the first equation of (2.2) has the form

$$\zeta = A(y)e^{-\sqrt{y}\eta} + B(y)e^{\sqrt{y}\eta} \qquad \left(\eta = \frac{(3k_0)^{1/2}x}{R_0}\right). \tag{2.3}$$

For a specific problem A and B can be determined, after which substitution of (2.3) in the second equation of (2.2) will make it possible to determine ξ and, hence, y(t).

We consider the following cases:

1. On a solid wall there is a layer of thickness h of uniformly distributed cavitation bubbles of initial radius R_0 at pressure p_0. At time t=0 a pressure P is instantaneously imposed on the layer boundary (x=h) and is subsequently maintained. We determine the pressure on the solid wall.

The boundary conditions for (2.3) are

$$\zeta = P - p_0\, y^{-3\gamma} \qquad \left(\eta = \eta^* = \frac{(3k_0)^{1/2} h}{R_0}\right),$$
$$\frac{\partial \zeta}{\partial \eta} = 0 \qquad (\eta = 0),$$

the last in view of symmetry.

Then, on the solid wall we obtain

$$p = P\,\frac{2e^{\sqrt{y}\eta^*}}{1 + e^{2\sqrt{y}\eta^*}} + p_0 y^{-3\gamma}\left(1 - \frac{2e^{\sqrt{y}\eta^*}}{1 + e^{2\sqrt{y}\eta^*}}\right). \tag{2.4}$$

To determine y we can use the second equation of (2.2) in its initial form (Lamb equation)

$$\rho_0 R_0^2\left(\frac{d^2 y}{dt^2} + \frac{3}{2}\left(\frac{dy}{dt}\right)^2\right) = p_0 y^{-3\gamma} - p, \tag{2.5}$$

Figure 2 shows the result of calculation of function p(t) for P=100 p_0 and η^*=3 from Eq. (2.4) and (2.5). Despite the fact that P=1 atm is imposed on the boundary, a pressure of several tens of atmospheres arises on the wall, as Fig. 2 shows. This confirms the conclusion in [5] to the effect that collapse of the cavitation bubbles on the plane leads not only to erosive damage to the surface, but also to generation by the bubble-containing medium of pressure pulses on the entire surface of the wall. Figures 3 and 4 show the maximum pressure p* (in atm) on the solid wall and the minimum radius y* (in dimensionless form) of the collapsed bubble as functions of η^*, where curves 1, 2, 3, 4 and 5 correspond to p_0=0.33, 0.5, 1, 1.4 and 10 (10^4 dyne/cm^2).

The function y* (η^*) practically determines one of the main parameters of pulsation of the "collective" bubble – the degree of compression; the second parameter – the collapse time t* (emission time for the first pulse in Fig. 2) – is determined by complete linearization of (2.2) and neglect of the pressure inside the bubble

$$t^* = 0.755\, R_0 \left(\frac{\rho_0}{P}\right)^{1/2} e^{1/2\,\eta^*}. \tag{2.6}$$

The agreement between the values of t* calculated from Eq. (2.6) and obtained by numerical solution of (2.2) is perfectly satisfactory.

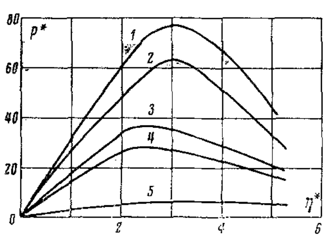

Fig. 3

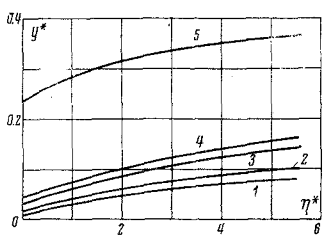

Fig. 4

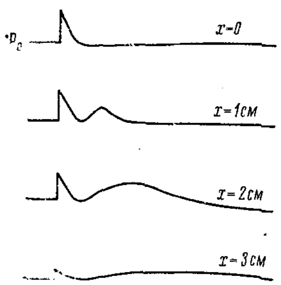

Fig. 5

2. In the case considered the solid wall can be replaced by an incompressible liquid, and the pressure on the front boundary can be specified as P(t) [or P(y) in the new variables]. The solution of (2.2) is given, as before, by Eq. (2.4) and (2.5).

Figure 5 shows calculation results for the shape of the pressure wave in the medium at the boundary of the incompressible liquid for bubble layers of length x=0, 1, 2, and 3 cm, respectively, and gas concentration $k_0=0.08$. The form of p(t) in Fig. 5 at x=0 gives the form of the initial shock wave (on the front boundary) with the following parameters: maximum amplitude − 10 atm, time of positive pressure phase − 10^{-4} sec.

With these parameters we carried out an experiment on shock-wave interaction with a bubble-containing medium. The experimental results are shown in Fig. 6. The frames in Fig. 6 correspond (as in Fig. 5) to the values x=0, 1, 2, 3 cm in descending order. A comparison of Figs. 5 and 6 reveals that (2.2) quite satisfactorily describes the nature of the interaction and takes into account the main factors: stratification of the shock wave and transmission of energy to the emission of the bubble-containing medium.

Equation (2.4) for the pressure in the bubble-containing medium indicates that the superscript

$$\eta^* = \frac{(3k_0)^{1/2}h}{R_0} = \frac{\Omega(R_0)h}{c_1} \tag{2.7}$$

plays the role of similarity criterion. By changing k_0, R_0, and h within a fixed η^* we obtain the same result. This similarity was confirmed experimentally. In fact, when t=0, y=1 and Eq. (2.4) gives the amplitude of the shock wave passing through the given layer.

Figure 7 shows the relationship between $P^0=(p-p_0)/P(t)$ for t=0 and η^*, which will be universal and independent of P when $P \gg p_0$. On the graph we have plotted the experimental data for $P^0(\eta^*)$ for different initial gas concentrations k_0 in the medium: 1) 0.004; 2) 0.02; 3) 0.06; 4) 0.08; 5) 0.10; 6) 0.15. The agreement is perfectly satisfactory.

3. The coefficients in Eq. (2.3) can also be determined for a semi-infinite bubble-containing medium on the assumption of a bounded solution at infinity. In this case, instead of (2.4), we have

$$p = Pe^{-\sqrt{y}\eta} + p_0 y^{-3\gamma}(1 - e^{-\sqrt{y}\eta}). \tag{2.8}$$

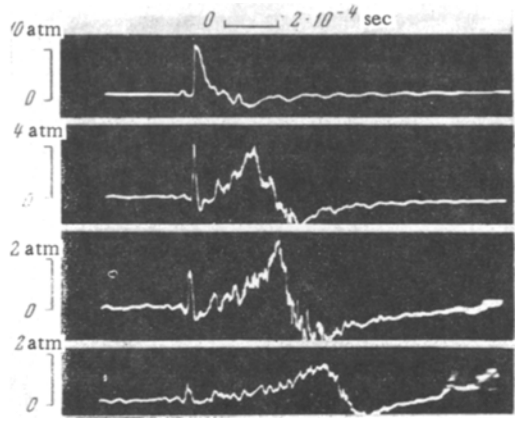

Fig. 6

For a shock wave of triangular profile with maximum amplitude 20 atm and with time of positive pressure phase 10^{-4} sec for $k_0=0.002$ and $R_0=0.4$ cm, we made a calculation from (2.8). The calculation results (curves 1-9) for the pressure distribution (in atm) in the medium for various fixed instants (1) 1.4; 2) 2.5; 3) 3.3; 4) 4.4; 5) 5.8; 6) 22.7; 7) 32.9; 8) 39.1; 9) 46.1 [10^{-4}] sec) are shown in Fig. 8. The figure shows that a wave with an amplitude which decreases exponentially with time is propagated with variable velocity through the medium.

After some time (due to the fact that the bubbles continue to pulsate) a compression wave arises again in the initial layers and is again propagated inside the layer.

3. The system of motion equations for a liquid containing gas bubbles can be written in a form different from (0.1): The Lamb equation can be replaced by the energy-conservation equation. The latter can be obtained from simple physical ideas of the nature of the compression-wave interaction with the bubble-containing medium. It is obvious that during propagation the wave must expend energy on alteration of the internal energy E_n of the gas and the kinetic energy T_n of the liquid due to the pulsating bubbles. We assume in this case that the change in the internal and kinetic energy of the liquid component is small.

The change in energy per unit volume of the medium can then be written as

$$\frac{\partial}{\partial x}\int_0^\infty pu\,dt = -(T_n + E_n) \qquad \left(E_n = -\int_{k_0}^{k} p_0\left(\frac{k_0}{k}\right)^\gamma dk\right). \tag{3.1}$$

The brackets contain the expression for the internal-energy change E_n of the gas per unit volume on the assumption that there are bubbles of only one kind and their number n per unit volume of the incompressible liquid is constant; for the change in kinetic energy we have

$$T_n = {}^3/_2\rho_0 k\,(dR\,/\,dt)^2 \qquad (k = n\,{}^4/_3\pi R^3).$$

On the other hand, the first integral in the bubble-pulsation equation expresses the energy balance

$$\frac{3}{2}\rho_0 k\left(\frac{dR}{dt}\right)^2 - \int_{k_0}^{k} p_0\left(\frac{k_0}{k}\right)^\gamma dk + \int_{k_0}^{k} p\,dk = 0. \tag{3.2}$$

Substituting the expression $-\int_{k_0}^{k} p\,dk$ from (3.2) in place of $(T_n + E_n)$ in (3.1) and changing the variables,

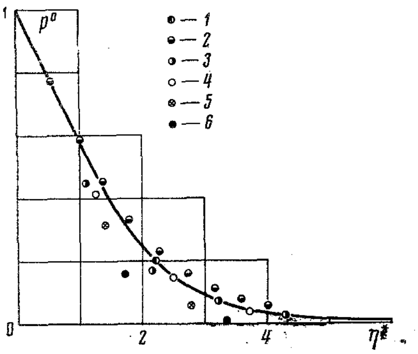

Fig. 7

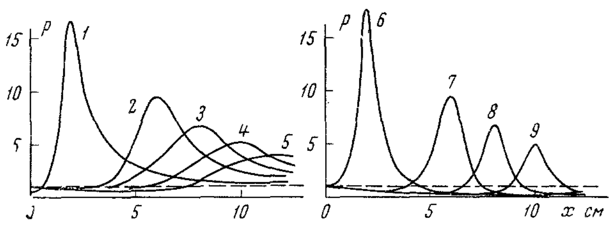

Fig. 8

we obtain

$$\frac{\partial (pu)}{\partial x} = p \frac{dk}{dt}. \quad (3.3)$$

The use of Eq. (3.3) instead of the Lamb equation of Eq. (0.1) leads to a system of first-order equations.

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